A generalized expansion method for nonlinear wave equations

2009 J. Phys. A: Math. Theor. 42 045207

(http://iopscience.iop.org/1751-8121/42/4/045207)

J. Phys. A: Math. Theor. **42** (2009) 045207 (23pp) doi:10.1088/1751-8113/42/4/045207

A generalized expansion method for nonlinear wave equations

Qun Lin[1], **Yong Hong Wu and Ryan Loxton**

Department of Mathematics and Statistics, Curtin University of Technology, Perth, Australia

E-mail: Q.Lin@curtin.edu.au

Received 18 September 2008, in final form 6 November 2008
Published 19 December 2008
Online at stacks.iop.org/JPhysA/42/045207

Abstract
A generalized Jacobian/exponential expansion method for finding the exact traveling wave solutions of a nonlinear partial differential equation is discussed. We use this method to construct many new, previously undiscovered exact solutions for the Boussinesq and modified KdV equations. We also apply it to the shallow long wave approximate equations. New solutions are deduced for this system of partial differential equations.

PACS numbers: 02.30.Jr, 04.20.Jb

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Solutions of partial differential equations have attracted significant interest in the literature. Exact traveling wave solutions, in particular, are useful both in practice and for verifying the accuracy and stability of popular numerical schemes such as the finite difference and finite element methods. By employing a computer algebra software such as Maple or Mathematica, the large amounts of tedious working required to verify candidate traveling wave solutions can be avoided. The capability and power of these softwares have increased dramatically over the past decade. Hence, a direct search for exact solutions is now much more viable.

Several effective direct search methods have been proposed in the literature. These include the tanh method [15, 16], exp-function method [6, 20], Jacobian elliptic function method [12, 18], Weierstrass's elliptic function method [17], reduction of order methods [9, 10], and cosh/sinh ansatz I–III method [19]. In this paper, we extend the generalized expansion method developed in [2, 3]. More specifically, we develop some new Jacobian elliptic and exponential solution classes for the same auxiliary ordinary differential equation (ODE) considered in these papers. The solutions of the ODE are then used to construct candidate traveling wave

[1] Author to whom any correspondence should be addressed.

solutions. Our new results ensure that, when applied to the classical Boussinesq and modified KdV equations, this generalized expansion method not only recovers all of the solutions reported in [6, 12, 18, 19, 22], but also discovers many new ones. Furthermore, this approach is flexible as well as powerful—it is easily adapted in section 6 to handle the system of shallow long wave approximate equations.

2. Preliminary results

The Jacobian elliptic functions are thoroughly discussed in [1, 5]. Since these special functions play an important role in the following, we will briefly introduce them here. We will also discuss some preliminary results that form the basis for our work in sections 3–6. Note that we will follow the usual convention and let i denote the complex number satisfying $\mathrm{i}^2 = -1$. Moreover, for the remainder of this paper, $m \in (0, 1)$ is arbitrary.

To begin with, consider the integral

$$\zeta = \int_0^{\rho} \frac{\mathrm{d}\eta}{\sqrt{1 - m^2 \sin^2(\eta)}}.$$

Here, the constant m is referred to as the modulus and the upper limit ρ is called the amplitude of ζ, which we denote as

$$\rho = \mathrm{am}(\zeta).$$

On this basis, the first three Jacobian elliptic functions are defined as

$$\mathrm{sn}(\zeta) := \sin[\mathrm{am}(\zeta)] = \sin(\rho),$$
$$\mathrm{cn}(\zeta) := \cos[\mathrm{am}(\zeta)] = \cos(\rho)$$

and

$$\mathrm{dn}(\zeta) := \sqrt{1 - m^2 \sin^2[\mathrm{am}(\zeta)]} = \sqrt{1 - m^2 \sin^2(\rho)}.$$

As $m \to 1$, we have

$$\mathrm{sn}(\zeta) \to \tanh(\zeta), \qquad \mathrm{cn}(\zeta) \to \mathrm{sech}(\zeta), \qquad \mathrm{dn}(\zeta) \to \mathrm{sech}(\zeta).$$

Similarly, as $m \to 0$,

$$\mathrm{sn}(\zeta) \to \sin(\zeta), \qquad \mathrm{cn}(\zeta) \to \cos(\zeta), \qquad \mathrm{dn}(\zeta) \to 1.$$

Nine additional Jacobian elliptic functions can be defined in terms of these first three—see [1, 5] for details.

In [2, 3], the following auxiliary ODE was introduced:

$$[F'(\xi)]^2 = q_0 + q_1 F(\xi) + q_2[F(\xi)]^2 + q_3[F(\xi)]^3 + q_4[F(\xi)]^4, \tag{2.1}$$

where $q_k, k = 0, \ldots, 4$, are given coefficients. Various solutions of ODE (2.1) were constructed using the Jacobian elliptic functions, and these results were exploited in the design of a systematic procedure for generating solutions of nonlinear partial differential equations. We will follow a similar approach in this paper. In our work, ODE (2.1) will be considered assuming $q_4 \neq 0$. We will need to determine more general solution classes of ODE (2.1) than those reported in [2, 3]. This is the motivation behind the preliminary results that follow.

Recall that m is an arbitrary real number satisfying $0 < m < 1$. With this in mind, for any (possibly complex) number γ, define the constants $p_{j,k}(\gamma)$, $j = 1, \ldots, 12$, $k = 0, \ldots, 4$, according to the following table.

Furthermore, let the functions $\varphi_{j,k}(\cdot,\gamma)$, $j = 1,\ldots,12$, $k = 1,\ldots,4$ be defined as follows:

$$\varphi_{1,1}(\xi,\gamma) = \frac{\mathrm{dn}(\xi)}{\gamma\,\mathrm{dn}(\xi)+1},$$

$$\varphi_{1,2}(\xi,\gamma) = \frac{\sqrt{1-m^2}}{\gamma\sqrt{1-m^2}+\mathrm{dn}(\xi)},$$

$$\varphi_{1,3}(\xi,\gamma) = \frac{\sqrt{m^2-1}\,\mathrm{sn}(\xi)}{\gamma\sqrt{m^2-1}\,\mathrm{sn}(\xi)+\mathrm{cn}(\xi)},$$

$$\varphi_{1,4}(\xi,\gamma) = \frac{\mathrm{cn}(\xi)}{\gamma\,\mathrm{cn}(\xi)+\mathrm{i}\,\mathrm{sn}(\xi)},$$

$$\varphi_{2,1}(\xi,\gamma) = \frac{\mathrm{sn}(\xi)}{\gamma\,\mathrm{sn}(\xi)+1},$$

$$\varphi_{2,2}(\xi,\gamma) = \frac{1}{\gamma+m\,\mathrm{sn}(\xi)},$$

$$\varphi_{2,3}(\xi,\gamma) = \frac{\mathrm{dn}(\xi)}{\gamma\,\mathrm{dn}(\xi)+m\,\mathrm{cn}(\xi)},$$

$$\varphi_{2,4}(\xi,\gamma) = \frac{\mathrm{cn}(\xi)}{\gamma\,\mathrm{cn}(\xi)+\mathrm{dn}(\xi)},$$

$$\varphi_{3,1}(\xi,\gamma) = \frac{\mathrm{cn}(\xi)}{\gamma\mathrm{cn}(\xi)+1},$$

$$\varphi_{3,2}(\xi,\gamma) = \frac{\sqrt{m^2-1}}{\gamma\sqrt{m^2-1}+m\,\mathrm{cn}(\xi)},$$

$$\varphi_{3,3}(\xi,\gamma) = \frac{\mathrm{dn}(\xi)}{\gamma\,\mathrm{dn}(\xi)+\mathrm{i}m\,\mathrm{sn}(\xi)},$$

$$\varphi_{3,4}(\xi,\gamma) = \frac{\sqrt{1-m^2}\mathrm{sn}(\xi)}{\gamma\sqrt{1-m^2}\,\mathrm{sn}(\xi)+\mathrm{dn}(\xi)},$$

$$\varphi_{4,1}(\xi,\gamma) = \frac{1}{\gamma+\mathrm{i}m\,\mathrm{sn}(\xi)+\mathrm{dn}(\xi)},$$

$$\varphi_{4,2}(\xi,\gamma) = \frac{\mathrm{dn}(\xi)}{\gamma\,\mathrm{dn}(\xi)+\mathrm{i}m\,\mathrm{cn}(\xi)+\sqrt{1-m^2}},$$

$$\varphi_{4,3}(\xi,\gamma) = \frac{\mathrm{sn}(\xi)}{\gamma\mathrm{sn}(\xi)+\mathrm{i}+\mathrm{i}\,\mathrm{cn}(\xi)},$$

$$\varphi_{4,4}(\xi,\gamma) = \frac{\mathrm{cn}(\xi)}{\gamma\,\mathrm{cn}(\xi)+\mathrm{i}\,\mathrm{dn}(\xi)+\sqrt{m^2-1}\mathrm{sn}(\xi)},$$

$$\varphi_{5,1}(\xi,\gamma) = \frac{1}{\gamma+m\,\mathrm{cn}(\xi)+\mathrm{dn}(\xi)},$$

$$\varphi_{5,2}(\xi,\gamma) = \frac{\mathrm{dn}(\xi)}{\gamma\,\mathrm{dn}(\xi)+m\sqrt{1-m^2}\mathrm{sn}(\xi)+\sqrt{1-m^2}},$$

$$\varphi_{5,3}(\xi,\gamma) = \frac{\operatorname{sn}(\xi)}{\gamma\operatorname{sn}(\xi) + \mathrm{i}\operatorname{dn}(\xi) + \mathrm{i}\operatorname{cn}(\xi)},$$

$$\varphi_{5,4}(\xi,\gamma) = \frac{\operatorname{cn}(\xi)}{\gamma\operatorname{cn}(\xi) + \sqrt{m^2-1} + \sqrt{m^2-1}\operatorname{sn}(\xi)},$$

$$\varphi_{6,1}(\xi,\gamma) = \frac{1}{\gamma + \mathrm{i}\operatorname{sn}(\xi) + \operatorname{cn}(\xi)},$$

$$\varphi_{6,2}(\xi,\gamma) = \frac{\operatorname{dn}(\xi)}{\gamma\operatorname{dn}(\xi) + \mathrm{i}\operatorname{cn}(\xi) + \sqrt{1-m^2}\operatorname{sn}(\xi)},$$

$$\varphi_{6,3}(\xi,\gamma) = \frac{m\operatorname{sn}(\xi)}{\gamma m\operatorname{sn}(\xi) + \mathrm{i} + \mathrm{i}\operatorname{dn}(\xi)},$$

$$\varphi_{6,4}(\xi,\gamma) = \frac{\mathrm{i}m\operatorname{cn}(\xi)}{\mathrm{i}\gamma m\operatorname{cn}(\xi) + \operatorname{dn}(\xi) + \sqrt{1-m^2}},$$

$$\varphi_{7,1}(\xi,\gamma) = \frac{\sqrt{1-m^2}[1+\operatorname{sn}(\xi)]}{\gamma\sqrt{1-m^2} + \sqrt{1-m^2}(\gamma+1)\operatorname{sn}(\xi) + \operatorname{dn}(\xi)},$$

$$\varphi_{7,2}(\xi,\gamma) = \frac{\operatorname{dn}(\xi) + \operatorname{cn}(\xi)}{\gamma\operatorname{dn}(\xi) + (\gamma+1)\operatorname{cn}(\xi) + 1},$$

$$\varphi_{7,3}(\xi,\gamma) = \frac{\sqrt{1-m^2}[1+m\operatorname{sn}(\xi)]}{\gamma m\sqrt{1-m^2}\operatorname{sn}(\xi) + \sqrt{1-m^2}(\gamma+1) + m\mathrm{i}\operatorname{cn}(\xi)},$$

$$\varphi_{7,4}(\xi,\gamma) = \frac{\operatorname{dn}(\xi) + m\operatorname{cn}(\xi)}{m\gamma\operatorname{cn}(\xi) + (\gamma+1)\operatorname{dn}(\xi) + \mathrm{i}m\operatorname{sn}(\xi)},$$

$$\varphi_{8,1}(\xi,\gamma) = \frac{\operatorname{dn}(\xi) + \sqrt{1-m^2}\operatorname{sn}(\xi)}{(1+\sqrt{1-m^2}\gamma)\operatorname{sn}(\xi) + \gamma\operatorname{dn}(\xi)},$$

$$\varphi_{8,2}(\xi,\gamma) = \frac{\sqrt{1-m^2}[\operatorname{cn}(\xi)+1]}{\sqrt{1-m^2}\gamma + \sqrt{1-m^2}\gamma\operatorname{cn}(\xi) + \operatorname{cn}(\xi)},$$

$$\varphi_{8,3}(\xi,\gamma) = \frac{\sqrt{1-m^2} + \mathrm{i}m\operatorname{cn}(\xi)}{1 + \sqrt{1-m^2}\gamma + \mathrm{i}\gamma m\operatorname{cn}(\xi)},$$

$$\varphi_{8,4}(\xi,\gamma) = \frac{\sqrt{1-m^2}\operatorname{dn}(\xi) + m\sqrt{m^2-1}\operatorname{sn}(\xi)}{\operatorname{dn}(\xi) + \gamma\sqrt{1-m^2}\operatorname{dn}(\xi) + m\sqrt{m^2-1}\gamma\operatorname{sn}(\xi)},$$

$$\varphi_{9,1}(\xi,\gamma) = \frac{\operatorname{sn}(\xi) + \sqrt{1-m^2}\operatorname{dn}(\xi)}{m\sqrt{2-m^2} + \sqrt{-m^4+m^2+1}\operatorname{cn}(\xi)},$$

$$\varphi_{9,2}(\xi,\gamma) = \frac{\operatorname{cn}(\xi) - 1 + m^2}{m\sqrt{2-m^2}\operatorname{dn}(\xi) + \sqrt{(-m^4+m^2+1)(1-m^2)}\operatorname{sn}(\xi)},$$

$$\varphi_{9,3}(\xi,\gamma) = \frac{\operatorname{dn}(\xi) + \mathrm{i}m(1-m^2)\operatorname{sn}(\xi)}{m^2\sqrt{2-m^2}\operatorname{cn}(\xi) + \sqrt{(-m^4+m^2+1)(m^2-1)}},$$

$$\varphi_{9,4}(\xi,\gamma) = \frac{1 + m\sqrt{m^2-1}\operatorname{cn}(\xi)}{m^2\sqrt{2-m^2}\operatorname{sn}(\xi) + \sqrt{m^4-m^2-1}\operatorname{dn}(\xi)},$$

$$\varphi_{10,1}(\xi,\gamma)=\frac{\mathrm{cn}(\xi)+\sqrt{1-m^2}\,\mathrm{dn}(\xi)}{m^2-1+\sqrt{m^4-m^2+1}\,\mathrm{cn}(\xi)},$$

$$\varphi_{10,2}(\xi,\gamma)=\frac{\mathrm{sn}(\xi)+\sqrt{1-m^2}}{\sqrt{1-m^2}\,\mathrm{dn}(\xi)+\sqrt{m^4-m^2+1}\mathrm{sn}(\xi)},$$

$$\varphi_{10,3}(\xi,\gamma)=\frac{1+m\sqrt{1-m^2}\mathrm{sn}(\xi)}{m\sqrt{m^2-1}\,\mathrm{cn}(\xi)+\sqrt{m^4-m^2+1}},$$

$$\varphi_{10,4}(\xi,\gamma)=\frac{\mathrm{dn}(\xi)+m\sqrt{1-m^2}\,\mathrm{cn}(\xi)}{\mathrm{i}(m^3-m)\mathrm{sn}(\xi)+\sqrt{m^4-m^2+1}\,\mathrm{dn}(\xi)},$$

$$\varphi_{11,1}(\xi,\gamma)=\frac{\mathrm{cn}(\xi)+\sqrt{1-m^2}\,\mathrm{dn}(\xi)}{m+\sqrt{m^4-m^2+1}\mathrm{sn}(\xi)},$$

$$\varphi_{11,2}(\xi,\gamma)=\frac{\sqrt{1-m^2}\mathrm{sn}(\xi)-1+m^2}{m\,\mathrm{dn}(\xi)+\sqrt{m^4-m^2+1}\,\mathrm{cn}(\xi)},$$

$$\varphi_{11,3}(\xi,\gamma)=\frac{\mathrm{i}[\mathrm{dn}(\xi)+m\sqrt{1-m^2}\,\mathrm{cn}(\xi)]}{m^2\mathrm{sn}(\xi)+\sqrt{m^4-m^2+1}},$$

$$\varphi_{11,4}(\xi,\gamma)=\frac{\sqrt{m^2-1}[1+m\sqrt{1-m^2}\mathrm{sn}(\xi)]}{m^2\,\mathrm{cn}(\xi)+\sqrt{m^4-m^2+1}\,\mathrm{dn}(\xi)},$$

$$\varphi_{12,1}(\xi,\gamma)=\frac{\mathrm{cn}(\xi)+1-m^2}{\sqrt{1-m^2}\,\mathrm{dn}(\xi)+\sqrt{1-m^4+m^2}\,\mathrm{cn}(\xi)},$$

$$\varphi_{12,2}(\xi,\gamma)=\frac{\mathrm{sn}(\xi)+\sqrt{1-m^2}\,\mathrm{dn}(\xi)}{\sqrt{1-m^2}+\sqrt{1-m^4+m^2}\mathrm{sn}(\xi)},$$

$$\varphi_{12,3}(\xi,\gamma)=\frac{1+m\sqrt{m^2-1}\,\mathrm{cn}(\xi)}{m\sqrt{1-m^2}\mathrm{sn}(\xi)+\sqrt{1-m^4+m^2}},$$

$$\varphi_{12,4}(\xi,\gamma)=\frac{\mathrm{dn}(\xi)+\mathrm{i}(m^3-m)\mathrm{sn}(\xi)}{m\sqrt{1-m^2}\,\mathrm{cn}(\xi)+\sqrt{1-m^4+m^2}\,\mathrm{dn}(\xi)}.$$

Through the lengthy calculation, we can readily verify the following result. Note that Maple can be used to help us for the calculation.

Theorem 1. *Let γ be arbitrary. Then, for each $j=1,\ldots,12$, ODE (2.1) with coefficients $q_k=p_{j,k}(\gamma)$, $k=0,\ldots,4$, has solutions $\varphi_{j,k}(\cdot,\gamma)$, $k=1,\ldots,4$.*

Remark 1. Theorem 1 can be generalized further. In fact, it remains valid even if $\mathrm{cn}(\xi)$, $\mathrm{sn}(\xi)$ and $\mathrm{dn}(\xi)$ are replaced, respectively, by $\pm\mathrm{cn}(\xi)$, $\pm\mathrm{sn}(\xi)$ and $\pm\mathrm{dn}(\xi)$ in the expressions for $\varphi_{j,k}$ given above.

In some cases, the solutions of ODE (2.1) can be used to generate additional solutions. This observation is furnished precisely in theorems 2 and 3. Again, Maple can be used to conveniently verify these results.

Theorem 2. *Suppose that φ is a solution of ODE (2.1) with coefficients $q_k=\hat{q}_k$, $k=0,\ldots,4$, where $\hat{q}_1=\hat{q}_3=0$, and $\hat{q}_0$, $\hat{q}_2$ and $\hat{q}_4$ are given constants such that $\hat{q}_0\neq 0$. Then,*

$$\pm\sqrt{\frac{\hat{q}_4}{\hat{q}_0}}\varphi+\frac{1}{\varphi}$$

Table 1. The definition of the constants $p_{j,k}(\gamma)$, $j = 1, \ldots, 12$, $k = 0, \ldots, 4$.

j	$p_{j,0}(\gamma)$	$p_{j,1}(\gamma)$	$p_{j,2}(\gamma)$	$p_{j,3}(\gamma)$	$p_{j,4}(\gamma)$
1	m^2-1	$4\gamma(1-m^2)$	$2-6\gamma^2+ 6\gamma^2m^2-m^2$	$2\gamma(2\gamma^2-2+ m^2-2\gamma^2m^2)$	$\gamma^4m^2+2\gamma^2-\gamma^4- 1-\gamma^2m^2$
2	1	-4γ	$6\gamma^2-1-m^2$	$2\gamma(1+m^2-2\gamma^2)$	$\gamma^4+m^2-\gamma^2-\gamma^2m^2$
3	$1-m^2$	$4\gamma(m^2-1)$	$2m^2-6\gamma^2m^2+ 6\gamma^2-1$	$2\gamma(2\gamma^2m^2- 2\gamma^2+1-2m^2)$	$2\gamma^2m^2+\gamma^4-m^2- \gamma^4m^2-\gamma^2$
4	$-\frac{1}{4}$	γ	$\frac{-3\gamma^2+1-2m^2}{2}$	$\gamma(2m^2+\gamma^2-1)$	$\frac{-\gamma^4-1-4\gamma^2m^2+2\gamma^2}{4}$
5	$-\frac{1}{4}$	γ	$\frac{1-3\gamma^2+m^2}{2}$	$\gamma(\gamma^2-1-m^2)$	$\frac{2\gamma^2+2m^2-\gamma^4-1+2\gamma^2m^2-m^4}{4}$
6	$-\frac{m^2}{4}$	γm^2	$\frac{m^2-3\gamma^2m^2-2}{2}$	$\gamma(\gamma^2m^2-m^2+2)$	$\frac{2\gamma^2m^2-\gamma^4m^2-m^2-4\gamma^2}{4}$
7	0	m^2-1	$3\gamma+2-3\gamma m^2-m^2$	$3\gamma^2m^2+2\gamma m^2- 3\gamma^2-4\gamma-1$	$\gamma(\gamma+1) (\gamma+1-\gamma m^2)$
8	0	$-2\sqrt{1-m^2}$	$6\sqrt{1-m^2}\gamma-4m^2+5$	$(8m^2-10)\gamma- (6\gamma^2+4)\sqrt{1-m^2}$	$(4\gamma+2\gamma^3)\sqrt{1-m^2}+ 1+(5-4m^2)\gamma^2$
9	$\frac{1}{4}$	0	$-m^2+\frac{1}{2}$	0	$\frac{1}{4}$
10	$\frac{m^2}{4(1-m^2)}$	$\frac{\sqrt{m^4-m^2+1}}{m^2-1}$	$\frac{2m^4-3m^2+4}{2(1-m^2)}$	$\frac{\sqrt{m^4-m^2+1}}{m^2-1}$	$\frac{m^2}{4(1-m^2)}$
11	$\frac{1-m^2}{4}$	0	$\frac{1+m^2}{2}$	0	$\frac{1-m^2}{4}$
12	$\frac{m^2(2-m^2)}{4(1-m^2)}$	$\frac{\sqrt{1-m^4+m^2}}{m^2-1}$	$\frac{m^4-4}{2(m^2-1)}$	$\frac{\sqrt{1-m^4+m^2}}{m^2-1}$	$\frac{m^2(2-m^2)}{4(1-m^2)}$

is a solution of ODE (2.1) with coefficients

$$q_0 = 8\hat{q}_4 \mp 4\hat{q}_2\sqrt{\frac{\hat{q}_4}{\hat{q}_0}}, \qquad q_1 = 0, \qquad q_2 = \hat{q}_2 \mp 6\hat{q}_0\sqrt{\frac{\hat{q}_4}{\hat{q}_0}}, \qquad q_3 = 0, \qquad q_4 = \hat{q}_0.$$

Theorem 3. *Suppose that φ is a solution of ODE (2.1) with coefficients $q_k = \hat{q}_k$, $k = 0, \ldots, 4$, where $\hat{q}_k$, $k = 0, \ldots, 4$, are given constants such that $\hat{q}_1 \neq 0$ and $\hat{q}_4 = \frac{\hat{q}_0\hat{q}_3^2}{\hat{q}_1^2}$. Then,*

$$\frac{\hat{q}_3}{\hat{q}_1}\varphi + \frac{1}{\varphi}$$

is a solution of ODE (2.1) with coefficients

$$q_0 = \frac{4\hat{q}_3(2\hat{q}_0\hat{q}_3 - \hat{q}_1\hat{q}_2)}{\hat{q}_1^2}, \qquad q_1 = -4\hat{q}_3, \qquad q_2 = \hat{q}_2 - \frac{6\hat{q}_0\hat{q}_3}{\hat{q}_1}, \qquad q_3 = \hat{q}_1, \qquad q_4 = \hat{q}_0.$$

Remark 2. From table 1 and theorem 1, the reader will note that, for any γ, theorem 3 can be invoked with $\varphi_{j,k}(\cdot, \gamma)$, $j \in \{10, 12\}$, $k = 1, \ldots, 4$.

We would also like to consider the non-Jacobian elliptic solutions of ODE (2.1). As such, to conclude this section, we present the following two results. Both can be proved easily via direct substitution.

Theorem 4. *Let a_{-1}, a_0, a_1 and b_0 be given constants such that $a_{-1} \neq 0$ and $a_0 \neq a_{-1}b_0$. Then,*

$$\frac{a_{-1}\,\mathrm{e}^{-\xi} + a_0 + a_1\,\mathrm{e}^{\xi}}{\mathrm{e}^{-\xi} + b_0 + \frac{a_1}{a_{-1}}\,\mathrm{e}^{\xi}}$$

is a solution of ODE (2.1) with coefficients

$$q_0 = -\frac{\left(4a_{-1}a_1 - a_0^2\right)a_{-1}^2}{(a_{-1}b_0 - a_0)^2}, \qquad q_1 = \frac{2a_{-1}\left(-a_0a_{-1}b_0 + 8a_{-1}a_1 - a_0^2\right)}{(a_{-1}b_0 - a_0)^2},$$

$$q_2 = \frac{a_{-1}^2b_0^2 + 4a_{-1}a_0b_0 - 24a_{-1}a_1 + a_0^2}{(a_{-1}b_0 - a_0)^2}, \qquad q_3 = \frac{2(8a_1 - a_{-1}b_0^2 - a_0b_0)}{(a_{-1}b_0 - a_0)^2},$$

$$q_4 = \frac{a_{-1}b_0^2 - 4a_1}{a_{-1}(a_{-1}b_0 - a_0)^2}.$$

Theorem 5. *Let a_{-1}, a_1, b_0 and b_1 be given constants such that $a_1 \neq b_1a_{-1}$ and $a_0 = \frac{b_0(a_{-1}b_1+a_1)\pm(a_{-1}b_1-a_1)\sqrt{b_0^2-4b_1}}{2b_1}$. Then,*

$$\frac{a_{-1}\,\mathrm{e}^{-\xi} + a_0 + a_1\,\mathrm{e}^{\xi}}{\mathrm{e}^{-\xi} + b_0 + b_1\,\mathrm{e}^{\xi}}$$

is a solution of ODE (2.1) with coefficients

$$q_0 = \frac{a_{-1}^2a_1^2}{(b_1a_{-1} - a_1)^2}, \qquad q_1 = \frac{-2a_{-1}a_1^2 - 2b_1a_{-1}^2a_1}{(b_1a_{-1} - a_1)^2},$$

$$q_2 = \frac{a_1^2 + 4a_{-1}b_1a_1 + a_{-1}^2b_1^2}{(b_1a_{-1} - a_1)^2}, \qquad q_3 = \frac{-2a_1b_1 - 2a_{-1}b_1^2}{(b_1a_{-1} - a_1)^2}, \qquad q_4 = \frac{b_1^2}{(b_1a_{-1} - a_1)^2}.$$

Note that additional solutions of ODE (2.1) can be constructed using Weierstrass' elliptic function. The reader is directed to [17] for more details.

3. A generalized expansion method

We will briefly outline a generalized expansion method for constructing traveling wave solutions. Similar procedures have been developed in [2, 3]. However, the new results given in the previous section ensure that our method yields many new solutions when applied to some classical partial differential equations. This will be clearly demonstrated in sections 4–6.

We consider the following nonlinear wave equation:

$$H(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \ldots) = 0, \tag{3.1}$$

where $u := u(x, t)$ is a real or complex-valued function, H is a given function involving powers of its arguments and the subscripts denote differentiation. We will consider candidate traveling wave solutions that take the form

$$u(x, t) = \tilde{u}(\xi) = \sum_{j=0}^{N} c_j[F(\xi)]^j, \tag{3.2}$$

where $\xi = \mu(x - \nu t)$, $\mu > 0$ is the wave number, ν is the traveling wave velocity, N is an integer, F is a non-trivial solution of ODE (2.1) with coefficients $q_k, k = 0, \ldots, 4$ and $c_j, j = 0, \ldots, N$ are constants with $c_N \neq 0$. Depending on the form of H, μ and ν will be determined or remain as free parameters.

Note that $\tilde{u}$ given by (3.2) is a polynomial function of F. Hence, it is readily seen that, for each integer $\kappa \geqslant 1$, $\tilde{u}^\kappa$ is also a polynomial in F. In this case, we use the degree notation $O(\cdot)$ to denote the index of the highest power of F. Thus,

$$O(\tilde{u}^\kappa) = N\kappa, \quad \kappa \geqslant 1. \tag{3.3}$$

The derivatives of F can be obtained by repeatedly differentiating both sides of (2.1). For example,

$$\begin{cases} F'' = \dfrac{q_1}{2} + q_2 F + \dfrac{3q_3}{2} F^2 + 2q_4 F^3, \\ F''' = (q_2 + 3q_3 F + 6q_4 F^2) F', \\ F'''' = \left(3q_0 q_3 + \dfrac{1}{2} q_1 q_2\right) + \left(q_2^2 + \dfrac{9}{2} q_1 q_3 + 12 q_0 q_4\right) F \\ \qquad + 15\left(\dfrac{1}{2} q_2 q_3 + q_1 q_4\right) F^2 + \left(20 q_2 q_4 + \dfrac{15}{2} q_3^2\right) F^3 + 30 q_3 q_4 F^4 + 24 q_4^2 F^5. \end{cases} \tag{3.4}$$

It is not difficult to show that only the even derivatives are polynomials in F. The odd derivatives also contain the terms of the form $F^j(F')$, where j is a non-negative integer. In this case, we define $O(F') = 2$ and so

$$O(F^j(F')) = j + 2, \qquad j \geqslant 0.$$

By differentiating (3.2), we can also deduce the derivatives of $\tilde{u}$. For example,

$$\begin{cases} \tilde{u}' = (c_1 + \cdots + N c_N F^{N-1}) F', \\ \tilde{u}'' = (c_1 + \cdots + N c_N F^{N-1}) F'' + [2c_2 + \cdots + N(N-1) c_N F^{N-2}](F')^2, \\ \tilde{u}''' = (c_1 + \cdots + N c_N F^{N-1}) F''' + 3[2c_2 + \cdots + N(N-1) c_N F^{N-2}] F' F'' \\ \qquad + [6c_3 + \cdots + N(N-1)(N-2) c_N F^{N-3}](F')^3, \\ \tilde{u}'''' = (c_1 + \cdots + N c_N F^{N-1}) F'''' \\ \qquad + 4[2c_2 + \cdots + N(N-1) c_N F^{N-2}] F' F''' \\ \qquad + 3[2c_2 + \cdots + N(N-1) c_N F^{N-2}](F'')^2 \\ \qquad + 6[6c_3 + \cdots + N(N-1)(N-2) c_N F^{N-3}](F')^2 F'' \\ \qquad + [24c_4 + \cdots + N(N-1)(N-2)(N-3) c_N F^{N-4}](F')^4, \end{cases} \tag{3.5}$$

where the derivatives of F are given in (2.1) and (3.4). Higher-order derivatives can be obtained similarly. Again, only the even derivatives of $\tilde{u}$ are polynomials in F. It is readily seen that

$$O(\tilde{u}^{(\kappa)}) = N + \kappa, \qquad \kappa \geqslant 1. \tag{3.6}$$

When $\tilde{u}$ is substituted into (3.1), the original partial differential equation in x and t is reduced to a nonlinear ODE in ξ. We will normally choose N so that the degrees of the highest-order derivative term and highest-order nonlinear term in this reduced ODE are balanced. However, this does not always result in an integral value for N. In this case, it is sometimes possible to proceed by letting $\tilde{u} = v^{\frac{1}{\tau}}$, where τ is the denominator of the fractional value of N (assuming that the denominator and the numerator have no common factors), and solving the resulting equation for v. This is illustrated in the following example.

Example 1. Consider the following Boussinesq-type equation:

$$u_{tt} - u_{xx} + u_{xxxx} + (u^5 - u^3)_{xx} = 0.$$

By letting $u(x, t) = \tilde{u}(\mu(x - \nu t))$, the above partial differential equation is reduced to the following ODE:

$$\nu^2 \tilde{u}'' - \tilde{u}'' + \mu^2 \tilde{u}'''' + (\tilde{u}^5 - \tilde{u}^3)'' = 0.$$

Integrating twice yields

$$\nu^2 \tilde{u} - \tilde{u} + \mu^2 \tilde{u}'' + \tilde{u}^5 - \tilde{u}^3 = 0. \tag{3.7}$$

Here, the highest-order nonlinear term is $\tilde{u}^5$ and the highest-order derivative term is $\tilde{u}''$. Balancing these two terms using (3.3) and (3.6) gives $5N = N+2$ or $N = \frac{1}{2}$. Setting $\tilde{u} = v^{\frac{1}{2}}$, (3.7) becomes

$$(\nu^2 - 1)v^2 + \frac{\mu^2}{4}[2vv'' - (v')^2] + v^4 - v^3 = 0. \tag{3.8}$$

Now, we can balance $(v')^2$ and v^4 to yield $N = 1$. Hence, we can search for traveling wave solutions of (3.8) which take the form $v(\mu(x-\nu t)) = c_0 + c_1F(\mu(x-\nu t))$, for constants c_0 and c_1. If such a v can be determined, then it is easy to derive $\tilde{u}$.

It is noted in example 1 that substituting $\tilde{u}$ into (3.1) yields a nonlinear ODE in ξ. When the derivatives of $\tilde{u}$ are substituted into this reduced ODE, we will obtain a linear combination of $F^j(F')^k$, where $j \geqslant 0$ is an integer and $k \in \{0,1\}$. If ν, μ, and c_j, $j = 0,\dots,N$, and q_k, $k = 0,\dots,4$ can be chosen to make each coefficient in this linear combination zero, then the resulting $\tilde{u}$ will satisfy the original partial differential equation (3.1). However, in this procedure, we sometimes end up with $c_j = 0$, $j = 0,\dots,N$ (we encounter this in section 6). In this case, we can use the following alternative solution form proposed in [2]:

$$\tilde{u}(\xi) = c_{0,0} + \sum_{j=1}^{N} \frac{c_{1,j}[F(\xi)]^j + c_{2,j}[F(\xi)]^{j-1}F'(\xi)}{[\theta F(\xi)+1]^j}, \tag{3.9}$$

where $c_{0,0}$, $c_{k,j}$, $k = 1,2$, $j = 1,\dots,N$ and θ are constants.

Note that each of the Jacobian elliptic solutions of ODE (2.1) reported in [2, 13, 14] can be written as a scalar multiple of some $\varphi_{j,k}(\cdot,0)$, $j \in \{1,\dots,6\}$, $k \in \{1,\dots,4\}$. Hence, by applying our expansion method with (3.2) and theorem 1 to a nonlinear partial differential equation, we can replicate every Jacobian elliptic solution obtained using the methods presented in [13, 14]. Applying our expansion method with (3.9) and theorem 1 to a nonlinear partial differential equation, we can obtain all Jacobian elliptic solutions obtained using the method presented in [2]. Similarly, each Jacobian elliptic solution of ODE (2.1) reported in [3, 4] with $\omega = 1$ can be written as a scalar multiple of some $\varphi_{j,k}(\cdot,0)$, $j \in \{1,\dots,6\}$, $k \in \{1,\dots,4\}$. It is also evident that, for the special case $\theta = 0$, using our expansion method with (3.9) and theorems 1 and 2, we can recover every Jacobian elliptic solution obtained using the method of [3, 21]. Hence, by virtue of the new results in section 2, our method is a significant generalization of the work reported in [2, 3, 13, 14].

4. Traveling wave solutions for the Boussinesq equation

Consider the well-known Boussinesq equation

$$u_{tt} = u_{xx} + u_{xxxx} + 3(u^2)_{xx}, \tag{4.1}$$

where $u := u(x,t)$ is a real-valued function. Various methods have been used to solve Boussinesq-type equations [7, 8, 11]. Here, the general expansion method will be used to derive new traveling wave solutions for (4.1). Letting $u(x,t) = \tilde{u}(\xi)$, where ξ is as defined in section 3, (4.1) becomes the following ODE:

$$\nu^2\tilde{u}'' = \tilde{u}'' + \mu^2\tilde{u}'''' + 3(\tilde{u}^2)''. \tag{4.2}$$

Balancing $(\tilde{u}^2)''$ and $\tilde{u}''''$ gives $2N+2 = N+4$ or $N = 2$. Hence, we will search for traveling wave solutions of the form

$$\tilde{u}(\xi) = c_0 + c_1F(\xi) + c_2[F(\xi)]^2, \tag{4.3}$$

where $c_2 \neq 0$, and F satisfies ODE (2.1) with coefficients $q_k, k = 0, \ldots, 4$. Substituting (4.3) into (4.2) and using (2.1) and (3.4)–(3.5), we obtain the following sufficient conditions for $\tilde{u}$ to satisfy (4.1):

$$\begin{cases} c_0 = \dfrac{3\mu^2 q_3^2 - 16\mu^2 q_4 q_2 - 4q_4 + 4q_4\nu^2}{24q_4}, \\ c_1 = -\mu^2 q_3, \\ c_2 = -2\mu^2 q_4, \\ q_1 = \dfrac{q_3\left(4q_2q_4 - q_3^2\right)}{8q_4^2}. \end{cases} \tag{4.4}$$

That is, if a solution F of ODE (2.1) with coefficients satisfying $q_1 = \frac{q_3(4q_2q_4-q_3^2)}{8q_4^2}$ and $q_4 \neq 0$ can be found, then

$$\tilde{u}(\xi) = \frac{3\mu^2 q_3^2 - 16\mu^2 q_4 q_2 - 4q_4 + 4q_4\nu^2}{24q_4} - \mu^2 q_3 F(\xi) - 2\mu^2 q_4 [F(\xi)]^2 \tag{4.5}$$

is a solution of the Boussinesq equation (4.1). Now, we generalize this solution form further. Note that if $q_1 = q_3 = 0$, then (4.5) reduces to

$$\tilde{u}(\xi) = \frac{\nu^2 - 1 - 4\mu^2 q_2}{6} - 2\mu^2 q_4 [F(\xi)]^2. \tag{4.6}$$

If $q_0 \neq 0$, then using theorem 2 with (4.6) gives the following solution form for equation (4.1):

$$\tilde{u}(\xi) = \frac{\nu^2 - 1 - 4\mu^2 q_2}{6} - 2\mu^2 q_4 [F(\xi)]^2 - \frac{2\mu^2 q_0}{[F(\xi)]^2}. \tag{4.7}$$

Furthermore, note that (4.5) can be rewritten as

$$\tilde{u}(\xi) = \frac{\mu^2 q_3^2}{4q_4} + \frac{\nu^2 - 4\mu^2 q_2 - 1}{6} - 2\mu^2 q_4 \left(F(\xi) + \frac{q_3}{4q_4}\right)^2. \tag{4.8}$$

The solution forms (4.7) and (4.8) provide motivation for the following, more general, candidate traveling wave solution:

$$\tilde{u}(\xi) = \frac{\mu^2 q_3^2}{4q_4} + \frac{\nu^2 - 4\mu^2 q_2 - 1}{6} - 2\mu^2 q_4 \left(F(\xi) + \frac{q_3}{4q_4}\right)^2 + d\left(F(\xi) + \frac{q_3}{4q_4}\right)^{-2}, \tag{4.9}$$

where d is a constant. By substituting (4.9) into (4.2), the value of d can be determined. We summarize our results in the form of the following theorem.

Theorem 6. *For each $j = 1, 2$, let $\varepsilon_j \in \{0, 1\}$. Suppose that F is a solution of ODE (2.1) with coefficients q_j, $j = 0, \ldots, 4$ satisfying $q_4 \neq 0$ and $q_1 = \frac{q_3(4q_2q_4-q_3^2)}{8q_4^2}$. Then, for any μ and ν,*

$$u(x,t) = \frac{\mu^2 q_3^2}{4q_4} + \frac{\nu^2 - 4\mu^2 q_2 - 1}{6} - 2\varepsilon_1 \mu^2 q_4 \left(F(\mu(x-\nu t)) + \frac{q_3}{4q_4}\right)^2 + \varepsilon_2 \frac{\mu^2(16q_3^2 q_2 q_4 - 5q_3^4 - 256q_4^3 q_0)}{128q_4^3\left(F(\mu(x-\nu t)) + \frac{q_3}{4q_4}\right)^2}$$

is a solution of the Boussinesq equation (4.1).

Remark 3. Note that the solution form given in theorem 6 includes both (4.5) and (4.7) as special cases.

Note that the coefficients of ODE (2.1) in theorem 5 satisfy the requirements of theorem 6. Thus, we can apply theorem 6 with the solutions reported in theorem 5 to obtain the following class of traveling wave solutions for the Boussinesq equation (4.1):

$$u_1(x,t) = \frac{\nu^2 - 1 + 2\mu^2}{6} - \varepsilon_1 \frac{\mu^2}{2}\left(\frac{-\mathrm{e}^{-\mu(x-\nu t)} \pm \sqrt{\lambda^2 - 4\vartheta} + \vartheta\,\mathrm{e}^{\mu(x-\nu t)}}{\mathrm{e}^{-\mu(x-\nu t)} + \lambda + \vartheta\,\mathrm{e}^{\mu(x-\nu t)}}\right)^2 - \varepsilon_2 \frac{\mu^2}{2}\left(\frac{\mathrm{e}^{-\mu(x-\nu t)} + \lambda + \vartheta\,\mathrm{e}^{\mu(x-\nu t)}}{-\mathrm{e}^{-\mu(x-\nu t)} \pm \sqrt{\lambda^2 - 4\vartheta} + \vartheta\,\mathrm{e}^{\mu(x-\nu t)}}\right)^2,$$

where for each $j = 1, 2, \varepsilon_j \in \{0, 1\}$, and λ, ϑ, μ and ν are arbitrary real constants such that $\vartheta \leqslant \lambda^2/4$.

It should be addressed here that the above class of solutions includes all of those obtained by combining theorems 4 and 6. Note that, for some cases, the denominators in the expression of u_1 can be equal to zero at certain points, and thus, such a solution is unbounded. For example, u_1 with $\varepsilon_1 = \varepsilon_2 = 1$ and $\vartheta \neq 0$ is unbounded. It is also noted that, for some cases, the solution u_1 is bounded. For instance, u_1 with $\varepsilon_1 = 1, \varepsilon_2 = 0, 0 \leqslant \vartheta \leqslant \lambda^2/4$ and $\lambda \geqslant 0$ is bounded. For the bounded case, clearly, the solution u_1 gives a single wave that moves in the x-direction with velocity ν and as $\mu(x - \nu t) \to \pm\infty, u_1(x,t) \to (\nu^2 - 1 + 2\mu^2)/6 - \mu^2(\varepsilon_1 + \varepsilon_2)/2$.

Choosing $\vartheta = 1$ and replacing λ by 2λ, u_1 becomes

$$u(x,t) = \frac{\nu^2 - 1 + 2\mu^2}{6} - \varepsilon_1 \frac{\mu^2}{2}\left(\frac{\sinh[\mu(x-\nu t)] \pm \sqrt{\lambda^2 - 1}}{\cosh[\mu(x-\nu t)] + \lambda}\right)^2 - \varepsilon_2 \frac{\mu^2}{2}\left(\frac{\cosh[\mu(x-\nu t)] + \lambda}{\sinh[\mu(x-\nu t)] \pm \sqrt{\lambda^2 - 1}}\right)^2, \tag{4.10}$$

where for each $j = 1, 2, \varepsilon_j \in \{0, 1\}$, and λ, μ and ν are arbitrary real constants such that $\lambda \geqslant 1$ or $\lambda \leqslant -1$.

Since u is a real-valued function, the arbitrary constants in u_1 are generally real. However, this is actually an unnecessary restriction—these constants can be complex provided that u_1 remains real. If μ is replaced by $\mathrm{i}\mu$ in (4.10), then we obtain another class of solutions:

$$u_2(x,t) = \frac{\nu^2 - 1 - 2\mu^2}{6} - \varepsilon_1 \frac{\mu^2}{2}\left(\frac{\sin[\mu(x-\nu t)] \pm \sqrt{1 - \lambda^2}}{\cos[\mu(x-\nu t)] + \lambda}\right)^2 - \varepsilon_2 \frac{\mu^2}{2}\left(\frac{\cos[\mu(x-\nu t)] + \lambda}{\sin[\mu(x-\nu t)] \pm \sqrt{1 - \lambda^2}}\right)^2,$$

where for each $j = 1, 2, \varepsilon_j \in \{0, 1\}$, and μ, ν and λ are arbitrary real constants such that $-1 \leqslant \lambda \leqslant 1$. Obviously, the solution u_2 is unbounded.

Remark 4. In [19], the solutions of (4.1) were obtained using the sinh/cosh ansatz I–II method, the sinh–cosh ansatz III method, the tanh method and the sine–cosine method. Each of these solutions is a special case of u_1 or u_2.

To apply theorem 1 in conjunction with theorem 6, we must choose γ so that the hypotheses of theorem 6 are satisfied. That is, γ should be chosen so that the following equation holds:

$$p_{j,1}(\gamma) = \frac{p_{j,3}(\gamma)(4p_{j,2}(\gamma)p_{j,4}(\gamma) - [p_{j,3}(\gamma)]^2)}{8[p_{j,4}(\gamma)]^2}.$$

Then, the corresponding solutions reported in theorem 1 can be used with theorem 6. Applying this procedure, we can obtain periodic solutions of Boussinesq equation (4.1) in terms of Jacobi elliptic functions. For each $j = 1, \ldots, 12$, choosing $q_k = p_{j,k}$, $k = 0, \ldots, 4$ and employing theorem 6 give solutions u_j, $j = 3, \ldots, 26$. Some of the solutions are listed below.

$$u_3(x,t) = \frac{\nu^2 - 1 + 4\mu^2(m^2+1)}{6} - 2\mu^2[\varepsilon_1 m^2 \operatorname{sn}^2(\xi) + \varepsilon_2 \operatorname{sn}^{-2}(\xi)],$$

$$u_4(x,t) = \frac{\nu^2 - 1 + 4\mu^2(m^2-2)}{6} + 2\mu^2\left[\varepsilon_1 \operatorname{dn}^2(\xi) + \frac{1-m^2}{\operatorname{dn}^2(\xi)}\right],$$

$$u_5(x,t) = \frac{\nu^2 - 1 + 4\mu^2(m^2-2)}{6} - 2\mu^2\left[\varepsilon_1 \frac{\operatorname{cn}^2(\xi)}{\operatorname{sn}^2(\xi)} + \frac{(1-m^2)\operatorname{sn}^2(\xi)}{\operatorname{cn}^2(\xi)}\right],$$

$$u_6(x,t) = \frac{\nu^2 - 1 + 4\mu^2(m^2+1)}{6} - 2\mu^2\left[m^2 \frac{\operatorname{cn}^2(\xi)}{\operatorname{dn}^2(\xi)} + \frac{\operatorname{dn}^2(\xi)}{\operatorname{cn}^2(\xi)}\right],$$

$$u_7(x,t) = \frac{\nu^2 - 1 - 4\mu^2(2m^2-1)}{6} - 2\mu^2\left[-m^2 \operatorname{cn}^2(\xi) + \frac{1-m^2}{\operatorname{cn}^2(\xi)}\right],$$

$$u_8(x,t) = \frac{\nu^2 - 1 - 4\mu^2(2m^2-1)}{6} - 2\mu^2\left[\frac{\operatorname{dn}^2(\xi)}{\operatorname{sn}^2(\xi)} + \frac{m^2(m^2-1)\operatorname{sn}^2(\xi)}{\operatorname{dn}^2(\xi)}\right],$$

$$u_9(x,t) = \frac{\nu^2 - 1 + 2\mu^2(2m^2-1)}{6} - \frac{\mu^2[1-\operatorname{cn}(\xi)]}{2[1+\operatorname{cn}(\xi)]},$$

where for each $j = 1, 2$, $\varepsilon_j \in \{0, 1\}$, and μ and ν are arbitrary real constants. The other Jacobi elliptic function solutions are listed in the appendix.

Remark 5. It follows from remark 1 that u_j, $j = 3, \ldots, 26$, still satisfy the Boussinesq equation (4.1) even if $\operatorname{cn}(\xi)$, $\operatorname{sn}(\xi)$ and $\operatorname{dn}(\xi)$ are replaced, respectively, by $\pm\operatorname{cn}(\xi)$, $\pm\operatorname{sn}(\xi)$ and $\pm\operatorname{dn}(\xi)$.

Remark 6. It is interesting to note that, for each $j \in \{3, \ldots, 26\}$, the solution u_j becomes a special case of u_1 as $m \to 1$. Similarly, as $m \to 0$, u_j becomes a special case of u_2.

Remark 7. The solution u_3 is identical to the solutions reported in [12, 22], and the solution u_9 is the same as the solution reported in [23] (for $c_0^2 = 1$, $\alpha = 1$ and $\beta = 3$). However, all of the other Jacobian elliptic function solutions are new solutions. Furthermore, if the candidate traveling wave solutions of the form (3.9) are considered and our new results in section 2 are applied, then many additional solutions can be obtained.

To show the physical insight of these solutions, here we take u_4 and u_7 as examples. Figure 1 shows the wave profile of the solution u_4 with $m = 0.99$, $\mu = 1$ and $\nu = -1$. Clearly, the solution is a periodic function describing the traveling of waves in the negative x-direction. Figure 2 shows the graph of the solution u_7 for $m = 0.9$, $\mu = 1$ and $\nu = -2$. Note that u_7 becomes infinity when $\operatorname{cn}(\mu(x - \nu t), m) = 0$, that is, $\mu(x - \nu t) = (2n+1)K$, where $K = \int_0^{\pi/2} \left(1 - m^2 \sin^2(s)\right)^{-1/2} \mathrm{d}s$ and $n = 0, \pm 1, \ldots$ For instance, in figure 2, u_7 becomes negative infinity when the point (x, t) is close to the lines $x + 2t = 2.280\,549\,138(2n+1)$, where $n = 0, \pm 1, \ldots$ It is also noted from the expression of the solutions u_3 with $\varepsilon_2 = 1$, $u_5, \ldots, u_9$ that these solutions are unbounded, since the denominator in the expression can be zero at certain points.

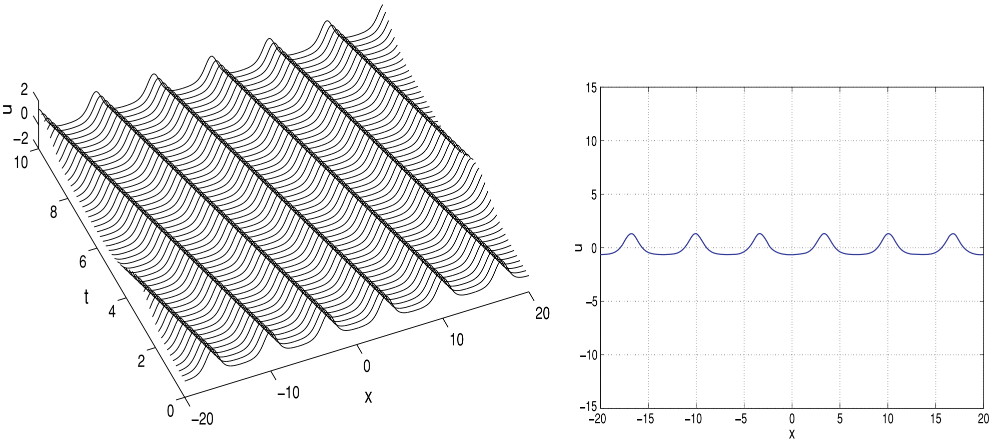

Figure 1. The plot of the solution u_4 to the Boussinesq equation (4.1) with $m = 0.99$, $\mu = 1$ and $\nu = -1$ and the initial status of u_4.

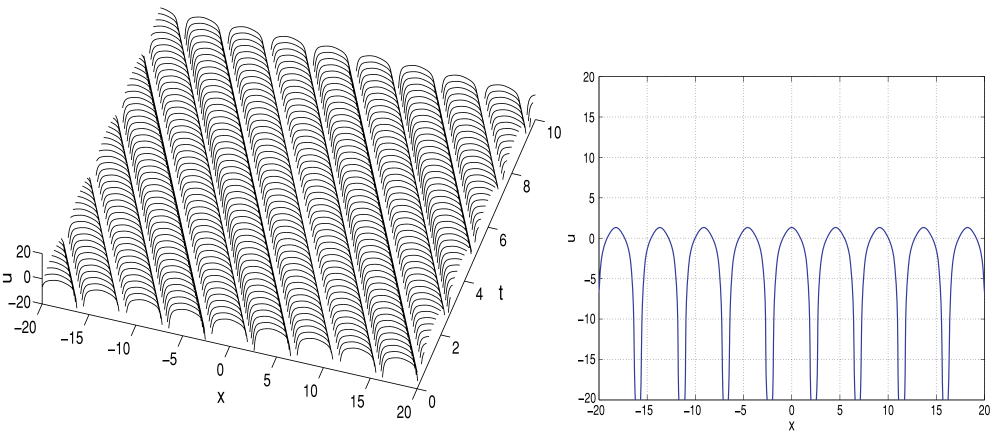

Figure 2. The plot of the solution u_7 to the Boussinesq equation (4.1) with $m = 0.9$, $\mu = 1$ and $\nu = -2$ and the initial status of u_7.

5. Traveling wave solutions for the modified KdV equation

We consider the following modified KdV equation:

$$u_t + u^2 u_x + u_{xxx} = 0, \tag{5.1}$$

where $u := u(x, t)$ is a complex-valued function. Letting $u(x, t) = \tilde{u}(\xi)$, where ξ is as defined in section 3, (5.1) is reduced to the ODE

$$-\nu\tilde{u}' + \tilde{u}^2\tilde{u}' + \mu^2\tilde{u}''' = 0. \tag{5.2}$$

Balancing $\tilde{u}^2\tilde{u}'$ and $\tilde{u}'''$ yields $N = 1$. Thus, we now consider candidate traveling wave solutions of the form

$$\tilde{u}(\xi) = c_0 + c_1 F(\xi),$$

where $c_1 \neq 0$, and F satisfies ODE (2.1) with coefficients $q_k, k = 0, \ldots, 4$. Substituting $\tilde{u}$ into (5.2), we obtain the following sufficient conditions for $\tilde{u}$ to satisfy (5.2):

$$\begin{cases} c_1^2 + 6\mu^2 q_4 = 0, \\ 2c_0c_1 + 3\mu^2 q_3 = 0, \\ -\nu + c_0^2 + \mu^2 q_2 = 0. \end{cases} \tag{5.3}$$

According to (5.3),

$$u(x,t) = \pm\mu\left(\frac{3q_3}{2\sqrt{-6q_4}} - \sqrt{-6q_4}F\left(\mu\left(x-\nu_1 t\right)\right)\right), \tag{5.4}$$

where $\nu_1 = \mu^2\big(q_2 - \frac{3q_3^2}{8q_4}\big)$ and μ is an arbitrary constant, is a solution of (5.1). Now, if $q_1 = q_3 = 0$ and $q_0 \neq 0$, then theorem 2 can be applied with (5.4) to give the following solution form of (5.1):

$$u(x,t) = \epsilon_1\mu\sqrt{-6q_0}\left(\epsilon_2\sqrt{\frac{q_4}{q_0}}F\left(\mu\left(x-\nu_2 t\right)\right) + \frac{1}{F\left(\mu\left(x-\nu_2 t\right)\right)}\right), \tag{5.5}$$

where $\epsilon_j = \pm 1, j = 1, 2, \nu_2 = \mu^2\big(q_2 - \epsilon_2 6q_0\sqrt{\frac{q_4}{q_0}}\big)$ and μ is an arbitrary constant. In addition, if $q_4 = \frac{q_0 q_3^2}{q_1^2}$ and $q_0 q_1 \neq 0$, then theorem 3 can be applied with (5.4) to yield another solution form of (5.1):

$$u(x,t) = \pm\mu\left[\frac{3q_1}{2\sqrt{-6q_0}} - \sqrt{-6q_0}\left(\frac{q_3}{q_1}F\left(\mu\left(x-\nu_3 t\right)\right) + \frac{1}{F\left(\mu\left(x-\nu_3 t\right)\right)}\right)\right], \tag{5.6}$$

where $\nu_3 = \mu^2\big(q_2 - \frac{6q_0q_3}{q_1} - \frac{3q_1^2}{8q_0}\big)$ and μ is an arbitrary constant.

We can apply theorem 4 with (5.4) to obtain the following class of traveling wave solutions of (5.1):

$$u_1(x,t) = \lambda + \frac{\frac{3\vartheta\mu^2}{\lambda}}{e^{-\mu(x-(\mu^2+\lambda^2)t)} + \vartheta + \frac{\vartheta^2(2\lambda^2+3\mu^2)}{8\lambda^2}\, e^{\mu(x-(\mu^2+\lambda^2)t)}},$$

where λ, ϑ and μ are arbitrary parameters such that $\lambda \neq 0$. It is noted that if λ, ϑ and μ are all real constants satisfying $\lambda\vartheta\mu \neq 0$, then u_1 describes a single wave traveling in the x-direction and $u_1(x,t) \to \lambda$, as $\mu(x - (\mu^2 + \lambda^2)t) \to \pm\infty$.

We can also apply theorem 5 with (5.6) to obtain another class of solutions of (5.1):

$$u_2(x,t) = \epsilon_1\frac{\sqrt{-6}\mu}{\sigma-1}\left(\frac{\sigma+1}{2} - \frac{\sigma\, e^{-\mu(x-\nu_4 t)} + \frac{1}{2}\lambda(\sigma+1) + \frac{1}{2}\epsilon_2\sqrt{\lambda^2-4\vartheta}(\sigma-1) + \vartheta\, e^{\mu(x-\nu_4 t)}}{e^{-\mu(x-\nu_4 t)} + \lambda + \vartheta\, e^{\mu(x-\nu_4 t)}}\right)$$
$$- \epsilon_1\frac{\sqrt{-6}\mu\sigma}{\sigma-1}\left(\frac{e^{-\mu(x-\nu_4 t)} + \lambda + \vartheta\, e^{\mu(x-\nu_4 t)}}{\sigma\, e^{-\mu(x-\nu_4 t)} + \frac{1}{2}\lambda(\sigma+1) + \frac{1}{2}\epsilon_2\sqrt{\lambda^2-4\vartheta}(\sigma-1) + \vartheta\, e^{\mu(x-\nu_4 t)}}\right),$$

where $\epsilon_j = \pm 1, j = 1, 2, \nu_4 = \frac{\mu^2(\sigma^2+10\sigma+1)}{2(\sigma-1)^2}$, and λ, ϑ and σ are arbitrary constants such that $\sigma \neq 1$. Note that u_1 is the same as solution (18) in [6], obtained using the exp-function method. However, u_2 is a new solution.

We can also obtain Jacobian elliptic solutions to the modified KdV equation (5.1) by combining theorem 1 with (5.4)–(5.6).

(1) For $k \in \{1, \ldots, 4\}$, $j \in \{1, \ldots, 12\}$ and γ arbitrary, (5.4) with $F = \varphi_{j,k}(\cdot, \gamma)$ and $q_l = p_{j,l}(\gamma), l = 0, \ldots, 4$, is a solution of (5.1).

(2) For $k \in \{1, \ldots, 4\}$ and $j \in \{1, 2, 3, 4, 5, 6, 9, 11\}$, (5.5) with $F = \varphi_{j,k}(\cdot, 0)$ and $q_l = p_{j,l}(0), l = 0, \ldots, 4$, is a solution of (5.1).

(3) For $k \in \{1, \ldots, 4\}$ and $j \in \{10, 12\}$, (5.6) with $F = \varphi_{j,k}(\cdot, 0)$ and $q_l = p_{j,l}(0), l = 0, \ldots, 4$, is a solution of (5.1).

Thus, we can obtain many Jacobian elliptic solutions of (5.1). To keep the details to minimum, we will not list them all here. Instead, we just select some of them to compare our results with those reported in [18, 22]. Note that our method can also be applied to the modified KdV equation considered in [18, 22].

Let γ be such that $\gamma \neq \pm 1$ and $\gamma \neq \pm m$. Choosing $q_k = p_{2,k}(\gamma), k = 0, \ldots, 4$, from (5.4), it follows that

$$u_3(x,t) = \mu\left\{\frac{3\gamma(1+m^2-2\gamma^2)}{\sqrt{-6(m^2-\gamma^2)(1-\gamma^2)}} - \frac{\sqrt{-6(m^2-\gamma^2)(1-\gamma^2)}\mathrm{sn}[\mu(x-\nu_5 t)]}{\gamma\,\mathrm{sn}[\mu(x-\nu_5 t)]+1}\right\},$$

$$u_4(x,t) = \mu\left\{\frac{3\gamma(1+m^2-2\gamma^2)}{\sqrt{-6(m^2-\gamma^2)(1-\gamma^2)}} - \frac{\sqrt{-6(m^2-\gamma^2)(1-\gamma^2)}}{\gamma + m\,\mathrm{sn}[\mu(x-\nu_5 t)]}\right\},$$

$$u_5(x,t) = \mu\left\{\frac{3\gamma(1+m^2-2\gamma^2)}{\sqrt{-6(m^2-\gamma^2)(1-\gamma^2)}} - \frac{\sqrt{-6(m^2-\gamma^2)(1-\gamma^2)}\,\mathrm{dn}[\mu(x-\nu_5 t)]}{\gamma\,\mathrm{dn}[\mu(x-\nu_5 t)] + m\,\mathrm{cn}[\mu(x-\nu_5 t)]}\right\},$$

$$u_6(x,t) = \mu\left\{\frac{3\gamma(1+m^2-2\gamma^2)}{\sqrt{-6(m^2-\gamma^2)(1-\gamma^2)}} - \frac{\sqrt{-6(m^2-\gamma^2)(1-\gamma^2)}\,\mathrm{cn}[\mu(x-\nu_5 t)]}{\gamma\,\mathrm{cn}[\mu(x-\nu_5 t)] + \mathrm{dn}[\mu(x-\nu_5 t)]}\right\},$$

where $\nu_5 = \mu^2\big[6\gamma^2 - 1 - m^2 - \frac{3\gamma^2(1+m^2-2\gamma^2)^2}{2(m^2-\gamma^2)(1-\gamma^2)}\big]$ and μ is an arbitrary constant, are solutions of (5.1). If γ is any real number such that $m < |\gamma| < 1$, then $u_k, k = 3, \ldots, 6$ are real and bounded. Moreover, if $\gamma = 0$, then according to (5.5) we can obtain the following two unbounded solutions:

$$u_7(x,t) = \sqrt{-6}\mu\left\{\pm m\,\mathrm{sn}[\mu(x-\nu_6 t)] + \frac{1}{\mathrm{sn}[\mu(x-\nu_6 t)]}\right\},$$

$$u_8(x,t) = \sqrt{-6}\mu\left\{\pm\frac{\mathrm{dn}[\mu(x-\nu_6 t)]}{\mathrm{cn}[\mu(x-\nu_6 t)]} + \frac{m\,\mathrm{cn}[\mu(x-\nu_6 t)]}{\mathrm{dn}[\mu(x-\nu_6 t)]}\right\},$$

where $\nu_6 = -\mu^2(1 \pm 6m + m^2)$ and μ is an arbitrary constant.

Similarly, if $q_k = p_{3,k}(\gamma), k = 0, \ldots, 4$, where γ is an arbitrary constant such that $\gamma \neq \pm 1$ and $\gamma \neq \pm\mathrm{i}\frac{m}{\sqrt{1-m^2}}$, then we get solutions of (5.1) as follows:

$$u_9(x,t) = \mu\left\{\frac{3\gamma(1-2m^2-2\gamma^2+2\gamma^2m^2)}{\sqrt{-6(\gamma^2m^2-m^2-\gamma^2)(1-\gamma^2)}} - \frac{\sqrt{-6(\gamma^2m^2-m^2-\gamma^2)(1-\gamma^2)}\,\mathrm{cn}[\mu(x-\nu_7 t)]}{\gamma\,\mathrm{cn}[\mu(x-\nu_7 t)]+1}\right\},$$

$$u_{10}(x,t) = \mu\left\{\frac{3\gamma(1-2m^2-2\gamma^2+2\gamma^2m^2)}{\sqrt{-6(\gamma^2m^2-m^2-\gamma^2)(1-\gamma^2)}} - \frac{\sqrt{6(\gamma^2m^2-m^2-\gamma^2)(1-\gamma^2)(1-m^2)}}{\gamma\sqrt{m^2-1} + m\,\mathrm{cn}[\mu(x-\nu_7 t)]}\right\},$$

$$u_{11}(x,t) = \mu\left\{\frac{3\gamma(1-2m^2-2\gamma^2+2\gamma^2m^2)}{\sqrt{-6(\gamma^2m^2-m^2-\gamma^2)(1-\gamma^2)}} - \frac{\sqrt{-6(\gamma^2m^2-m^2-\gamma^2)(1-\gamma^2)}\,\mathrm{dn}[\mu(x-\nu_7 t)]}{\gamma\,\mathrm{dn}[\mu(x-\nu_7 t)] + \mathrm{i}m\,\mathrm{sn}[\mu(x-\nu_7 t)]}\right\},$$

$$u_{12}(x,t) = \mu\left\{\frac{3\gamma(1-2m^2-2\gamma^2+2\gamma^2m^2)}{\sqrt{-6(\gamma^2m^2-m^2-\gamma^2)(1-\gamma^2)}} - \frac{\sqrt{-6(\gamma^2m^2-m^2-\gamma^2)(1-\gamma^2)(1-m^2)}\,\mathrm{sn}[\mu(x-\nu_7t)]}{\gamma\sqrt{1-m^2}\,\mathrm{sn}[\mu(x-\nu_7t)]+\mathrm{dn}[\mu(x-\nu_7t)]}\right\},$$

where $\nu_7 = \mu^2\left[2m^2-6\gamma^2m^2+6\gamma^2-1-\frac{3\gamma^2(1-2m^2-2\gamma^2+2\gamma^2m^2)^2}{2(m^2\gamma^2-m^2-\gamma^2)(1-\gamma^2)}\right]$ and μ is an arbitrary constant. Moreover, if $\gamma = 0$, then we have the unbounded solutions

$$u_{13}(x,t) = \mu\sqrt{6(m^2-1)}\left\{\pm\frac{m}{\sqrt{m^2-1}}\,\mathrm{cn}[\mu(x-\nu_8t)] + \frac{1}{\mathrm{cn}[\mu(x-\nu_8t)]}\right\},$$

$$u_{14}(x,t) = \mu\sqrt{6(1-m^2)}\left\{\mp\frac{1}{\sqrt{m^2-1}}\frac{\mathrm{dn}[\mu(x-\nu_8t)]}{\mathrm{sn}[\mu(x-\nu_8t)]} + m\frac{\mathrm{sn}[\mu(x-\nu_8t)]}{\mathrm{dn}[\mu(x-\nu_8t)]}\right\},$$

where $\nu_8 = \mu^2(2m^2-1\pm 6m\sqrt{m^2-1})$ and μ is an arbitrary constant.

If $q_k = p_{6,k}(\gamma), k = 0,\ldots,4$, where γ is an arbitrary constant such that $m^2\gamma^4+m^2+4\gamma^2-2m^2\gamma^2 \neq 0$, then we can obtain other four solutions of (5.1)

$$u_{15}(x,t) = \mu\left\{\frac{3\gamma(\gamma^2m^2-m^2+2)}{\sqrt{6(\gamma^4m^2+m^2+4\gamma^2-2\gamma^2m^2)}} - \frac{\sqrt{6(\gamma^4m^2+m^2+4\gamma^2-2\gamma^2m^2)}}{2\gamma+\mathrm{i}2\mathrm{sn}[\mu(x-\nu_9t)]+2\,\mathrm{cn}[\mu(x-\nu_9t)]}\right\},$$

$$u_{16}(x,t) = \mu\left\{\frac{3\gamma(\gamma^2m^2-m^2+2)}{\sqrt{6(\gamma^4m^2+m^2+4\gamma^2-2\gamma^2m^2)}} - \frac{\sqrt{6(\gamma^4m^2+m^2+4\gamma^2-2\gamma^2m^2)}\,\mathrm{dn}[\mu(x-\nu_9t)]}{2\gamma\,\mathrm{dn}[\mu(x-\nu_9t)]+\mathrm{i}2\,\mathrm{cn}[\mu(x-\nu_9t)]+2\sqrt{1-m^2}\mathrm{sn}[\mu(x-\nu_9t)]}\right\},$$

$$u_{17}(x,t) = \mu\left\{\frac{3\gamma(\gamma^2m^2-m^2+2)}{\sqrt{6(\gamma^4m^2+m^2+4\gamma^2-2\gamma^2m^2)}} - \frac{\sqrt{6(\gamma^4m^2+m^2+4\gamma^2-2\gamma^2m^2)}m\,\mathrm{sn}[\mu(x-\nu_9t)]}{2\gamma m\,\mathrm{sn}[\mu(x-\nu_9t)]+\mathrm{i}2+\mathrm{i}2\,\mathrm{dn}[\mu(x-\nu_9t)]}\right\},$$

$$u_{18}(x,t) = \mu\left\{\frac{3\gamma(\gamma^2m^2-m^2+2)}{\sqrt{6(\gamma^4m^2+m^2+4\gamma^2-2\gamma^2m^2)}} - \frac{\sqrt{6(\gamma^4m^2+m^2+4\gamma^2-2\gamma^2m^2)}\mathrm{i}m\,\mathrm{cn}[\mu(x-\nu_9t)]}{\mathrm{i}2\gamma m\,\mathrm{cn}[\mu(x-\nu_9t)]+2\,\mathrm{dn}[\mu(x-\nu_9t)]+2\sqrt{1-m^2}}\right\},$$

where $\nu_9 = \mu^2\left[\frac{m^2-3\gamma^2m^2-2}{2}+\frac{3\gamma^2(\gamma^2m^2-m^2+2)^2}{2(m^2\gamma^4+m^2+4\gamma^2-2m^2\gamma^2)}\right]$ and μ is an arbitrary constant. Furthermore, choosing $\gamma = 0$ yields that, for any μ,

$$u_{19}(x,t) = \sqrt{-6}\mu m\,\mathrm{sn}[\mu x+\mu^3(m^2+1)t],$$

$$u_{20}(x,t) = \sqrt{-6}\mu\frac{1}{\mathrm{sn}[\mu x+\mu^3(m^2+1)t]},$$

$$u_{21}(x,t) = \sqrt{-6}\mu m\frac{\mathrm{cn}[\mu x+\mu^3(m^2+1)t]}{\mathrm{dn}[\mu x+\mu^3(m^2+1)t]},$$

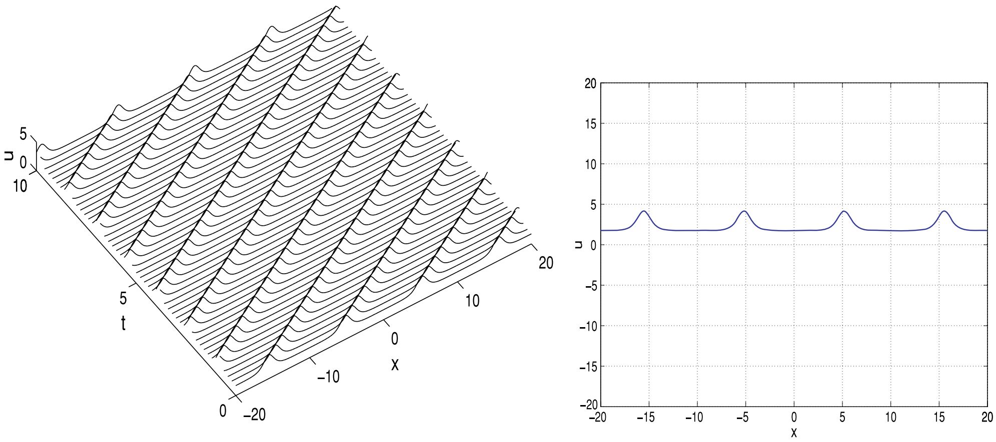

Figure 3. The plot of the solution u_6 to the modified KdV equation (5.1) with $m = 0.95$, $\mu = 1$ and $\gamma = 0.96$ and the initial status of u_6.

$$u_{22}(x,t) = \sqrt{-6}\mu \frac{\mathrm{dn}[\mu x + \mu^3(m^2+1)t]}{\mathrm{cn}[\mu x + \mu^3(m^2+1)t]},$$

$$u_{23}(x,t) = \sqrt{6}\mu m\, \mathrm{cn}[\mu x - \mu^3(2m^2-1)t],$$

$$u_{24}(x,t) = \mu\sqrt{6(m^2-1)}\frac{1}{\mathrm{cn}[\mu x - \mu^3(2m^2-1)t]},$$

$$u_{25}(x,t) = \mu\sqrt{-6}\frac{\mathrm{dn}[\mu x - \mu^3(2m^2-1)t]}{\mathrm{sn}[\mu x - \mu^3(2m^2-1)t]},$$

$$u_{26}(x,t) = \mu m\sqrt{6(1-m^2)}\frac{\mathrm{sn}[\mu x - \mu^3(2m^2-1)t]}{\mathrm{dn}[\mu x - \mu^3(2m^2-1)t]},$$

are solutions of (5.1).

Remark 8. It follows from remark 1 that u_j, $j = 3, \ldots, 26$, still satisfy (5.1) even if $\mathrm{cn}(\cdot)$, $\mathrm{sn}(\cdot)$ and $\mathrm{dn}(\cdot)$ are replaced, respectively, by $\pm\mathrm{cn}(\cdot)$, $\pm\mathrm{sn}(\cdot)$ and $\pm\mathrm{dn}(\cdot)$.

Remark 9. If $\gamma = 0$, then u_3, u_9 and u_{15} are the same as the solutions reported in [18] (with $a = 1$ and $b = 1$), and u_3, u_4 and u_7 are the same as those reported in [22] (for $\alpha = 1$ and $\beta = 1$). However, all of the other Jacobian elliptic solutions are new. More new solutions can be obtained if the solution form (3.9) is used.

To demonstrate the physical insight of the new solutions, we take u_6 as an example. By choosing $m = 0.95$ and $\mu = 1$, the wave profiles of the solution u_6 for two different values of γ, $\gamma = 0.96$ and $\gamma = -0.96$ are displayed in figures 3 and 4, respectively. Clearly, in both cases, the solutions describe the traveling of waves in the x-direction. Different values of γ yield different wave shapes.

6. Traveling wave solutions for the shallow long wave approximate equations

In this section, we will apply the method discussed in section 3 to a system of partial differential equations. Consider the shallow long wave approximate equations

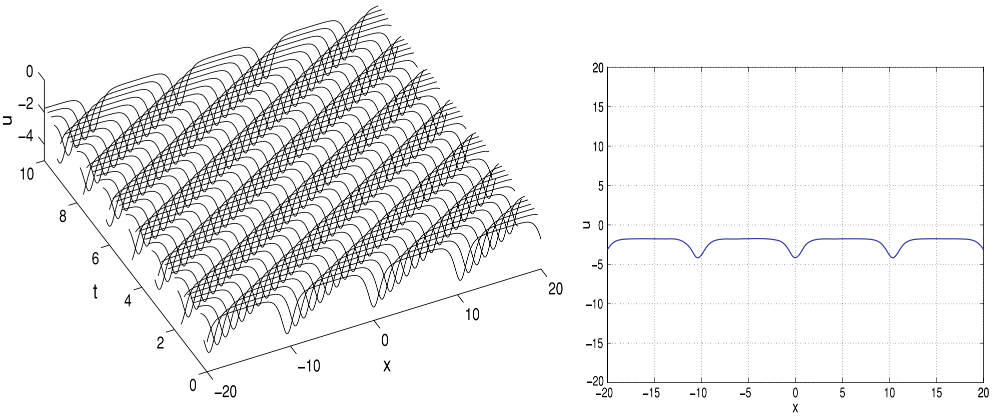

Figure 4. The plot of the solution u_6 to the modified KdV equation (5.1) with $m = 0.95$, $\mu = 1$ and $\gamma = -0.96$ and the initial status of u_6.

$$\begin{cases} u_t - uu_x - v_x + \frac{1}{2}u_{xx} = 0, \\ v_t - vu_x - uv_x - \frac{1}{2}v_{xx} = 0, \end{cases} \tag{6.1}$$

where $u := u(x,t)$ is the horizontal velocity of the water and $v := v(x,t)$ is the height that deviates from the equilibrium position of the water. Substituting $u(x,t) = \tilde{u}(\xi)$ and $v(x,t) = \tilde{v}(\xi)$, where ξ is as defined previously, into (6.1) and balancing the highest-order derivative and nonlinear terms, we obtain $N_u = 1$ and $N_v = 2$. If candidate traveling wave solutions are chosen according to (3.2), then all of the coefficients are required to be zero. Accordingly, we will use the more general form (3.9) and consider candidate solutions

$$\begin{cases} \tilde{u}(\xi) = \hat{c}_{0,0} + \dfrac{\hat{c}_{1,1}F(\xi) + \hat{c}_{2,1}F'(\xi)}{\theta F(\xi) + 1}, \\ \tilde{v}(\xi) = \tilde{c}_{0,0} + \dfrac{\tilde{c}_{1,1}F(\xi) + \tilde{c}_{2,1}F'(\xi)}{\theta F(\xi) + 1} + \dfrac{\tilde{c}_{1,2}F^2(\xi) + \tilde{c}_{2,2}F(\xi)F'(\xi)}{(\theta F(\xi) + 1)^2}, \end{cases} \tag{6.2}$$

where F satisfies ODE (2.1) with coefficients $q_k, k = 0, \ldots, 4$. By substituting (6.2) into (6.1), we can ascertain the following sufficient conditions for $\tilde{u}$ and $\tilde{v}$ to satisfy the shallow long wave approximate equations (6.1):

$$\mu = \pm\frac{\hat{c}_{1,1}}{\alpha} \qquad \hat{c}_{0,0} = -\nu + \frac{-4q_0\hat{c}_{1,1}\theta^3 + 3q_1\hat{c}_{1,1}\theta^2 - 2q_2\hat{c}_{1,1}\theta + q_3\hat{c}_{1,1}}{4\alpha^2},$$

$$\hat{c}_{2,1} = 0, \qquad \tilde{c}_{0,0} = \frac{\hat{c}_{1,1}^2}{16\alpha^4}\big[12q_0q_1\theta^5 - 8q_0^2\theta^6 - \big(12q_0q_2 + 3q_1^2\big)\theta^4 + (16q_0q_3 + 4q_1q_2)\theta^3 - (24q_0q_4 + 6q_1q_3)\theta^2 + 12q_1q_4\theta + q_3^2 - 4q_2q_4\big],$$

$$\tilde{c}_{1,1} = \frac{\hat{c}_{1,1}^2(4\theta^3q_0 - 3\theta^2q_1 + 2\theta q_2 - q_3)}{4\alpha^2}, \qquad \tilde{c}_{1,2} = -\frac{\hat{c}_{1,1}^2}{2},$$

$$\tilde{c}_{2,1} = \pm\frac{\hat{c}_{1,1}^2}{2\alpha}, \qquad \tilde{c}_{2,2} = \mp\frac{\hat{c}_{1,1}^2\theta}{2\alpha},$$

where $\alpha = \sqrt{q_0\theta^4 - q_1\theta^3 + q_2\theta^2 - q_3\theta + q_4}$ and $\theta, \nu, \hat{c}_{1,1}$ are arbitrary constants. Note that these requirements are the same as those reported in [2]. Note also that there are no conditions restricting the choice of coefficients $q_k, k = 0, \ldots, 4$, of ODE (2.1). Using $\varphi_{j,k}(\cdot, 0), j = 1, \ldots, 6, k = 1, \ldots, 4$, from theorem 1, we can reproduce the same Jacobian

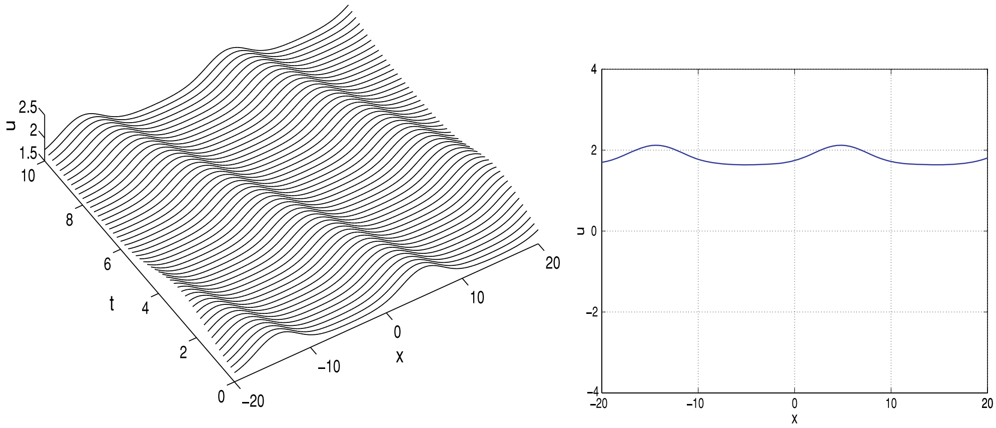

Figure 5. The plot of the solution u_1 of the shallow long wave approximate equations (6.1) with $m = 0.99$, $\nu = -2$ and $\vartheta = \gamma = 1$ and the initial status of u_1.

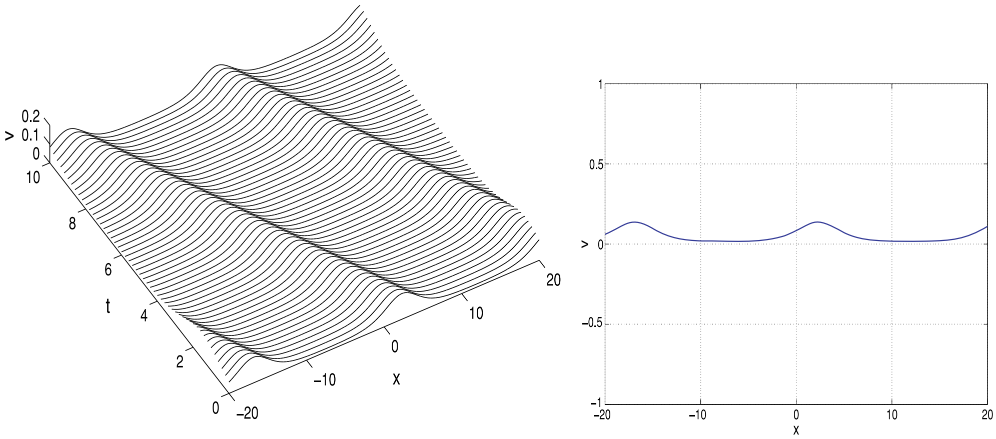

Figure 6. The plot of the solution v_1 to the shallow long wave approximate equations (6.1) with $m = 0.99$, $\nu = -2$ and $\vartheta = \gamma = 1$ and the initial status of v_1.

elliptic solutions of (6.1) reported in [2]. We also can deduce many new solutions by applying theorems 1–3. These solutions cannot be obtained using the results in [2]. For example, choosing $\theta = 0$ and $q_j = p_{7,j}(\gamma)$, $j = 0, \ldots, 4$, we can obtain the following solutions for the shallow long wave approximate equations (6.1):

$$u_j(x,t) = -\nu + \frac{\vartheta\beta}{4\alpha^2} + \vartheta\varphi_{7,j}\big(\mu(x-\nu t)\big), \qquad j = 1\ldots,4,$$

$$v_j(x,t) = -\vartheta^2\left\{\frac{\eta}{16\alpha^4} + \frac{\beta}{4\alpha^2}\varphi_{7,j}(\mu(x-\nu t)) - \frac{1}{2\alpha}\varphi'_{7,j}(\mu(x-\nu t)) + \frac{1}{2}\varphi^2_{7,j}(\mu(x-\nu t))\right\},$$
$$j = 1\ldots,4,$$

where $\varphi_{7,j}$, $j = 1,\ldots,4$, are as defined in section 2, $\alpha = \sqrt{\gamma^3(1-m^2)+\gamma^2(2-m^2)+\gamma}$, $\beta = \gamma^2(3m^2-3)+\gamma(2m^2-4)-1$, $\eta = \gamma^4(3m^4-6m^2+3)+\gamma^3(4m^4-12m^2+8)+\gamma^2(6-6m^2)-1$, $\mu = \vartheta/\alpha$, and ν, γ, ϑ and m ($m \in (0,1)$) are arbitrary. For the other solutions, we leave it to the reader.

To show the physical insight of these solutions, we take the solution (u_1, v_1) as an example. Figures 5 and 6 display the graphs of u_1 and v_1 with $m = 0.99$, $\nu = -2$ and $\vartheta = \gamma = 1$. Clearly, the solution describes the propagation of waves with horizontal velocity u_1 along the negative x-direction.

7. Conclusion

In this paper, we have presented a generalized expansion method for generating traveling wave solutions of nonlinear partial differential equations. This method has been successfully applied to the Boussinesq equation, the modified KdV equation and the shallow long wave approximate equations, and many new results have been obtained. For each equation investigated, we are able to replicate solutions previously derived in the literature, and discover many new ones. Extensions to two- and three-dimensional partial differential equations are possible. Other nonlinear partial differential equations can be tackled if an appropriate transformation can be found. For example, in [6], the transformation $u = \ln v$ was applied to the Dodd–Bullough–Mikhailov equation to yield a nonlinear partial differential equation involving powers of v and its derivatives.

Acknowledgments

The authors wish to acknowledge the reviewers for their comments and suggestions which have led to significant improvement of the paper.

Appendix

Some solutions of the Boussinesq equation (4.1), derived from this work, have been given in section 4, and the rest are listed below.

$$u_{10}(x,t) = \frac{\nu^2 - 1 + 2\mu^2(2m^2-1)}{6} - \frac{\mu^2[\mathrm{dn}(\xi) - \sqrt{1-m^2}\,\mathrm{sn}(\xi)]}{2[\mathrm{dn}(\xi) + \sqrt{1-m^2}\,\mathrm{sn}(\xi)]},$$

$$u_{11}(x,t) = \frac{\nu^2 - 1 - 2\mu^2(m^2+1)}{6} + \frac{\mu^2(1-m^2)[1 - m\,\mathrm{sn}(\xi)]}{2[1 + m\,\mathrm{sn}(\xi)]},$$

$$u_{12}(x,t) = \frac{\nu^2 - 1 - 2\mu^2(m^2+1)}{6} - \frac{\mu^2(1-m^2)[1 - \mathrm{sn}(\xi)]}{2[1 + \mathrm{sn}(\xi)]},$$

$$u_{13}(x,t) = \frac{\nu^2 - 1 - 2\mu^2(m^2+1)}{6} - \frac{\mu^2(1-m^2)[\mathrm{dn}(\xi) - \mathrm{cn}(\xi)]}{2[\mathrm{dn}(\xi) + \mathrm{cn}(\xi)]},$$

$$u_{14}(x,t) = \frac{\nu^2 - 1 - 2\mu^2(m^2+1)}{6} + \frac{\mu^2(m^2-1)[m\,\mathrm{cn}(\xi) - \mathrm{dn}(\xi)]}{2[m\,\mathrm{cn}(\xi) + \mathrm{dn}(\xi)]},$$

$$u_{15}(x,t) = \frac{\nu^2 - 1 - 2\mu^2(m^2-2)}{6} - \frac{\mu^2 m^2[\mathrm{dn}(\xi) - \sqrt{1-m^2}]}{2[\mathrm{dn}(\xi) + \sqrt{1-m^2}]},$$

$$u_{16}(x,t) = \frac{\nu^2 - 1 - 2\mu^2(m^2-2)}{6} - \frac{\mu^2 m^2[1 - \mathrm{dn}(\xi)]}{2[1 + \mathrm{dn}(\xi)]},$$

$$u_{17}(x,t) = \frac{\nu^2 - 1 - \mu^2(2m^2 + 12m + 2)}{6} + \frac{\mu^2(m-1)^2}{2}\left\{\left[\frac{1-\sqrt{m}\,\mathrm{sn}(\xi)}{1+\sqrt{m}\,\mathrm{sn}(\xi)}\right]^2 + \varepsilon_1\left[\frac{1+\sqrt{m}\,\mathrm{sn}(\xi)}{1-\sqrt{m}\,\mathrm{sn}(\xi)}\right]^2\right\},$$

$$u_{18}(x,t) = \frac{\nu^2 - 1 - \mu^2(2m^2 + 12m + 2)}{6} + \frac{\mu^2(m-1)^2}{2}\left\{\left[\frac{\mathrm{dn}(\xi)-\sqrt{m}\,\mathrm{cn}(\xi)}{\mathrm{dn}(\xi)+\sqrt{m}\,\mathrm{cn}(\xi)}\right]^2 + \varepsilon_1\left[\frac{\mathrm{dn}(\xi)+\sqrt{m}\,\mathrm{cn}(\xi)}{\mathrm{dn}(\xi)-\sqrt{m}\,\mathrm{cn}(\xi)}\right]^2\right\},$$

$$u_{19}(x,t) = \frac{\nu^2 - 1 - \mu^2(2m^2 - 4 + 12\sqrt{1-m^2})}{6} - \frac{\mu^2(1+\sqrt{1-m^2})^2}{2} \times\left\{\left[\frac{\mathrm{cn}(\xi)-\sqrt[4]{1-m^2}\,\mathrm{sn}(\xi)}{\mathrm{cn}(\xi)+\sqrt[4]{1-m^2}\,\mathrm{sn}(\xi)}\right]^2 + \varepsilon_1\left[\frac{\mathrm{cn}(\xi)+\sqrt[4]{1-m^2}\,\mathrm{sn}(\xi)}{\mathrm{cn}(\xi)-\sqrt[4]{1-m^2}\,\mathrm{sn}(\xi)}\right]^2\right\},$$

$$u_{20}(x,t) = \frac{\nu^2 - 1 - \mu^2(2m^2 - 4 - 12\sqrt{1-m^2})}{6} - \frac{\mu^2(1-\sqrt{1-m^2})^2}{2}\left\{\left[\frac{\mathrm{dn}(\xi)-\sqrt[4]{1-m^2}}{\mathrm{dn}(\xi)+\sqrt[4]{1-m^2}}\right]^2 + \varepsilon_1\left[\frac{\mathrm{dn}(\xi)+\sqrt[4]{1-m^2}}{\mathrm{dn}(\xi)-\sqrt[4]{1-m^2}}\right]^2\right\},$$

$$u_{21}(x,t) = \frac{\nu^2 - 1 + 2\mu^2(2m^2-1)}{6} - \frac{\mu^2}{2}\left[\frac{\mathrm{sn}(\xi)+\sqrt{1-m^2}\,\mathrm{dn}(\xi)}{m\sqrt{2-m^2}+\sqrt{-m^4+m^2+1}\,\mathrm{cn}(\xi)}\right]^2 - \frac{\mu^2}{2}\varepsilon_1\left[\frac{m\sqrt{2-m^2}+\sqrt{-m^4+m^2+1}\,\mathrm{cn}(\xi)}{\mathrm{sn}(\xi)+\sqrt{1-m^2}\,\mathrm{dn}(\xi)}\right]^2,$$

$$u_{22}(x,t) = \frac{\nu^2 - 1 + 2\mu^2(2m^2-1)}{6} - \frac{\mu^2}{2}\left[\frac{\mathrm{cn}(\xi)+(m^2-1)}{m\sqrt{2-m^2}\,\mathrm{dn}(\xi)+\sqrt{(-m^4+m^2+1)(1-m^2)}\,\mathrm{sn}(\xi)}\right]^2 - \varepsilon_1\frac{\mu^2}{2}\left[\frac{m\sqrt{2-m^2}\,\mathrm{dn}(\xi)+\sqrt{(-m^4+m^2+1)(1-m^2)}\,\mathrm{sn}(\xi)}{\mathrm{cn}(\xi)+(m^2-1)}\right]^2,$$

$$u_{23}(x,t) = \frac{\nu^2 - 1 - \mu^2(m^2+1)}{6} + \frac{\mu^2(m^2-1)}{2}\left[\frac{\mathrm{cn}(\xi)+\sqrt{1-m^2}\,\mathrm{dn}(\xi)}{m+\sqrt{m^4-m^2+1}\,\mathrm{sn}(\xi)}\right]^2 + \varepsilon_1\frac{\mu^2(m^2-1)}{2}\left[\frac{m+\sqrt{m^4-m^2+1}\,\mathrm{sn}(\xi)}{\mathrm{cn}(\xi)+\sqrt{1-m^2}\,\mathrm{dn}(\xi)}\right]^2,$$

$$u_{24}(x,t) = \frac{\nu^2 - 1 - \mu^2(m^2+1)}{6} - \frac{\mu^2(1-m^2)^2}{2}\left[\frac{\mathrm{sn}(\xi)+\sqrt{1-m^2}}{m\,\mathrm{dn}(\xi)+\sqrt{m^4-m^2+1}\,\mathrm{cn}(\xi)}\right]^2 - \varepsilon_1\frac{\mu^2}{2}\left[\frac{m\,\mathrm{dn}(\xi)+\sqrt{m^4-m^2+1}\,\mathrm{cn}(\xi)}{\mathrm{sn}(\xi)+\sqrt{1-m^2}}\right]^2,$$

$$u_{25}(x,t) = \frac{\nu^2 - 1 - \mu^2(m^2+1)}{6} - \frac{\mu^2(m^2-1)}{2}\left[\frac{\mathrm{dn}(\xi) + \sqrt{1-m^2}\,\mathrm{cn}(\xi)}{m^2\,\mathrm{sn}(\xi) + \sqrt{m^4-m^2+1}}\right]^2$$

$$- \varepsilon_1 \frac{\mu^2(m^2-1)}{2}\left[\frac{m^2\,\mathrm{sn}(\xi) + \sqrt{m^4-m^2+1}}{\mathrm{dn}(\xi) + \sqrt{1-m^2}\,\mathrm{cn}(\xi)}\right]^2,$$

$$u_{26}(x,t) = \frac{\nu^2 - 1 - \mu^2(m^2+1)}{6} + \frac{\mu^2(1-m^2)^2}{2}\left[\frac{1 + m\sqrt{1-m^2}\,\mathrm{sn}(\xi)}{m^2\,\mathrm{cn}(\xi) + \sqrt{m^4-m^2+1}\,\mathrm{dn}(\xi)}\right]^2$$

$$+ \varepsilon_1 \frac{\mu^2}{2}\left[\frac{m^2\,\mathrm{cn}(\xi) + \sqrt{m^4-m^2+1}\,\mathrm{dn}(\xi)}{1 + m\sqrt{1-m^2}\,\mathrm{sn}(\xi)}\right]^2,$$

where for each $j = 1, 2, \varepsilon_j \in \{0, 1\}$, and μ and ν are arbitrary real constants. Note that the solutions $u_j, j \in \{11, 14, 15, 16, 17, 18, 25, 26\}$, are bounded, while the solutions $u_j, j \in \{12, 13, 19, 20, 21, 22, 23, 24\}$, are unbounded.

References

[1] Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Washington, DC: National Bureau of Standards) (tenth printing)

[2] Chen Y and Wang Q 2006 A new elliptic equation rational expansion method and its application to the shallow long wave approximate equations *Appl. Math. Comput.* **173** 1163–82

[3] El-Sabbagh M F and Ali A T 2008 New generalized Jacobi elliptic function expansion method *Commun. Nonlinear Sci. Numer. Simul.* **13** 1758–66

[4] Fan E 2002 Multiple travelling wave solutions of nonlinear evolution equations using a unified algebraic method *J. Phys. A: Math. Gen.* **35** 6853–72

[5] Gradshteyn I S and Ryzhik I M 1980 *Table of Integrals, Series, and Products* (San Diego, CA: Academic)

[6] He J H and Wu X H 2006 Exp-function method for nonlinear wave equations *Chaos Solitons Fractals* **30** 700–8

[7] Lai S and Wu Y H 2003 The asymptotic solution of the Cauchy problem for a generalized Boussinesq equation *Discrete Continuous Dyn. Syst. Ser.* B **3** 401–8

[8] Lai S, Wu Y H and Yang X 2004 The global solution of an initial boundary value problem for the damped Boussinesq equation *Commun. Pure Appl. Anal.* **3** 319–28

[9] Lai S, Wu Y H and Wiwatanapataphee B 2008 On exact travelling wave solutions for two types of nonlinear $K(n, n)$ equations and a generalized KP equation *J. Comput. Appl. Math.* **212** 291–9

[10] Lai S, Wu Y H and Zhou Y 2008 Some physical structures for the (2 + 1)-dimensional Boussinesq water equation with positive and negative exponents *Comput. Math. Appl.* **56** 339–45

[11] Lin Q, Wu Y H, Loxton R and Lai S 2008 Linear B-spline finite element method for the improved Boussinesq equation *J. Comput. Appl. Math.* (doi:10.1016/j.cam.2008.05.049)

[12] Liu S, Fu Z, Liu S and Zhao Q 2001 Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations *Phys. Lett.* A **289** 69–74

[13] Liu J, Yang L and Yang K 2004 Nonlinear transformation and Jacobi elliptic function solutions of nonlinear equations *Chaos Solitons Fractals* **20** 1157–64

[14] Lv X, Lai S and Wu Y H 2007 An auxiliary equation technique and exact solutions for a nonlinear Klein–Gordon equation *Chaos Solitons Fractals* (doi:10.1016/j.chaos.2007.11.013)

[15] Malfliet W 1996 The tanh method: I. Exact solutions of nonlinear evolution and wave equations *Phys. Scr.* **54** 563–8

[16] Malfliet W 2004 The tanh method: a tool for solving certain classes of nonlinear evolution and wave equations *J. Comput. Appl. Math.* **164–165** 529–41

[17] Nickel J 2007 Elliptic solutions to a generalized BBM equation *Phys. Lett.* A **364** 221–6

[18] Shen S and Pan Z 2003 A note on the Jacobi elliptic function expansion method *Phys. Lett.* A **308** 143–8

[19] Wazwaz A 2006 New hyperbolic schemes for reliable treatment of Boussinesq equation *Phys. Lett.* A **358** 409–13

[20] Wu X H and He J H 2007 Solitary solutions, periodic solutions and compacton-like solutions using the Exp-function method *Comput. Math. Appl.* **54** 966–86

[21] Yan Z 2003 Jacobi elliptic function solutions of nonlinear wave equations via the new sinh-Gordon equation expansion method *J. Phys.* A **36** 1961–72

[22] Zhang H 2007 Extended Jacobi elliptic function expansion method and its applications *Commun. Nonlinear Sci. Numer. Simul.* **12** 627–35

[23] Zhang H 2007 New exact Jacobi elliptic function solutions for some nonlinear evolution equations *Chaos Solitons Fractals* **32** 653–60